Problem Set 1 #13

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1 Problem Description

The expression of A_D^2 that we derived, given in Eq. I.28, is proportional to the Lorentzian function,

$$f_L(\omega) = \frac{1}{\pi}\frac{\gamma}{(\omega-\omega_r)^2+\gamma^2} \tag{1}$$

a) Show that $\int_{-\infty}^{\infty} f_L(\omega)d\omega = 1$

b) Use MATLAB to plot $f_L(\omega)$ vs. $\omega-\omega_r$, assuming that $\gamma\omega_r = 1/100, 1/10\sqrt{10}, 1/10$

c) We stated that the expression for A_D in Eq. I.28 is a good approximation to the exact expression for A_D in Eq. I.21. Compare these expressions by plotting $A(\omega) = (\gamma/\omega_r)A_D(\omega)$ vs. $x = (\omega-\omega_r)/\gamma$ for both these expressions for $\gamma/\omega_r = 1/2, 1/10, 1/100$.

2 Solution

2.1 Part A

The first step is in this integration is to notice the similarity that the integral of $f_L(\omega)$ has with the trig function $\arctan(x)$. That is,

$$\arctan(x) = \int \frac{1}{x^2+1}dx \tag{2}$$

$$\int f_L(\omega) = \int \frac{1}{\pi}\frac{\gamma}{(\omega-\omega_r)^2+\gamma^2} \tag{3}$$

To get (3) in the same form as (2), we divide $f_L(\omega)$ through by γ^2, which gives

$$f_L(\omega) = \frac{1}{\pi}\frac{1}{\gamma}\frac{1}{[(\omega-\omega_r)/\gamma]^2+1} \tag{4}$$

Integrating (4) from $-\infty$ to ∞ with respect to ω gives

$$\int_{-\infty}^{\infty} f_L(\omega)d\omega = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{\gamma}\frac{1}{[(\omega-\omega_r)/\gamma]^2+1}d\omega \tag{5}$$

In order to use (2), we need to simplify (5) by performing a u substitution. This substitution will take the form of

$$u = \frac{\omega-\omega_r}{\gamma}, du = \frac{d}{d\omega}(\frac{\omega-\omega_r}{\gamma}) = \frac{1}{\gamma} \tag{6}$$

Plugging (6) back into (5), we get

$$\int_{-\infty}^{\infty} f_L(\omega)d\omega = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{u^2+1}du \tag{7}$$

Notice that (7) is almost identical to (2). Pulling the $1/\pi$ term outside the integral, integrating the rest, and then undoing the u substitution, (7) gives

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{u^2+1}du = \frac{1}{\pi}\arctan(u)|_{-\infty}^{\infty} = \frac{1}{\pi}\arctan(\frac{\omega-\omega_r}{\gamma})|_{-\infty}^{\infty} \tag{8}$$

Since ω is the variable of interest and is going to $\pm\infty$, ω_r and γ can be considered too small to be relevant and therefore can be discarded. (7) then becomes

$$\int_{-\infty}^{\infty} f_L(\omega)d\omega = \frac{1}{\pi}\arctan(\omega)|_{-\infty}^{\infty} \tag{9}$$

The $\arctan(x)$ function goes to $-\pi/2$ as x goes to $-\infty$ and $\pi/2$ as x goes to ∞, so evaluating (9) with these limits gives

$$\int_{-\infty}^{\infty} f_L(\omega)d\omega = \frac{1}{\pi}\arctan(\omega)|_{-\infty}^{\infty} = \frac{1}{\pi}(\frac{\pi}{2}-(-\frac{\pi}{2})) = \frac{1}{\pi}(\pi) = 1 \tag{10}$$

Therefore the integral of $\int_{-\infty}^{\infty} f_L(\omega)d\omega = 1$

2.2 Part B

The starting equation is $f_L(\omega)$ as used in Part A above. We need to change this equation to be a function of $\omega - \omega_r$ instead of ω. Since the relationship $\gamma\omega_r$ is known, by picking an arbitrary value for either γ or ω_r all unknowns can be eliminated from the equation. We can set a new variable $a = \gamma\omega_r$ so that $\gamma = a/\omega_r$. For simplicity's sake, we will set the value of ω_r to 1, so that the unknown variable γ can be substituted out for the known variables $a = \gamma\omega_r$ and ω_r. This approach yields the following change to $f_L(\omega)$:

$$f_L(\omega) = \frac{1}{\pi}\frac{\gamma}{(\omega-\omega_r)^2+\gamma^2} = \frac{1}{\pi}\frac{(a/\omega_r)}{(\omega-\omega_r)^2+(a/\omega_r)^2} \tag{11}$$

Once $f_L(\omega)$ is in this form, it can be plotted with respect to $\omega - \omega_r$. The following MATLAB code and plot shows this process for the three different values of $\gamma\omega_r$ that are presented in the problem.

```
%Variable List
%g_omr_1 = first value of gamma * omega_r
%g_omr_2 = second value of gamma * omega_r
%g_omr_3 = third value of gamma * omega_r
%omega_r = arbitrary value of omega_r
%omega_diff = range of omega - omega_r for solution to be plotted against
%a = temporary value of g_omr_1/2/3 used in solution
%f_L_1 = f_L solution with a set to g_omr_1
%f_L_2 = f_L solution with a set to g_omr_2
%f_L_3 = f_L solution with a set to g_omr_3

%Solution
g_omr_1 = 1/100;
g_omr_2 = 1/(10*sqrt(10));
g_omr_3 = 1/10;
%arbitrary value of omega_r, set to 1 for simplicity
omega_r = 1;
%arbitrary range of omega_diff, set to -100 to 100 to be significantly
%larger than omega_r
omega_diff = -100:1/100:100;
%f_L_1/2/3 calculations
a = g_omr_1;
f_L_1 = (1 / pi)*((a / omega_r) ./ ((omega_diff).^2 + (a / omega_r)^2));
a = g_omr_2;
f_L_2 = (1 / pi)*((a / omega_r) ./ ((omega_diff).^2 + (a / omega_r)^2));
a = g_omr_3;
f_L_3 = (1 / pi)*((a / omega_r) ./ ((omega_diff).^2 + (a / omega_r)^2));
```

Figure 1: Code to Generate Equations for each Relationship

```
%Plot Solution
figure(1)
%plot f_L_1/2/3 solutions
plot(omega_diff,[f_L_1; f_L_2; f_L_3])
%set x and y limits
xlim([-0.5 0.5])
ylim([0 35])
%set tick label interpreter to LaTeX
set(gca,'TickLabelInterpreter','LaTeX')
%label x and y axis and title
xlabel('$\omega - \omega_{r}$','interpreter','LaTeX')
ylabel('$L(\omega)$','interpreter','LaTeX')
title('$L(\omega)$ versus $(\omega-\omega_{r})$','interpreter','LaTex',...
    'fontname','Times New Roman')
%set x and y axis tick marks and tick labels
xticks([-0.5:0.1:0.5])
xticklabels({'$-0.5$','$-0.4$','$-0.3$','$-0.2$','$-0.1$','$0$',...
    '$0.1$','$0.2$','$0.3$','$0.4$','$0.5$'});
yticks([0:5:35])
yticklabels({'$0$','$5$','$10$','$15$','$20$','$25$','$30$','$35$'})
%turn on axis for all sides
box on
%set legend and remove box around it
lgd = legend('$\gamma/\omega_r=1/100$','$\gamma/\omega_r=1/10\sqrt10$',...
    '$\gamma/\omega_r=1/10$');
set(lgd,'interpreter','LaTeX')
legend('boxoff')
%save as pdf
saveas(gcf,'13b.pdf','pdf')
```

Figure 2: Code to Plot Generated Equations

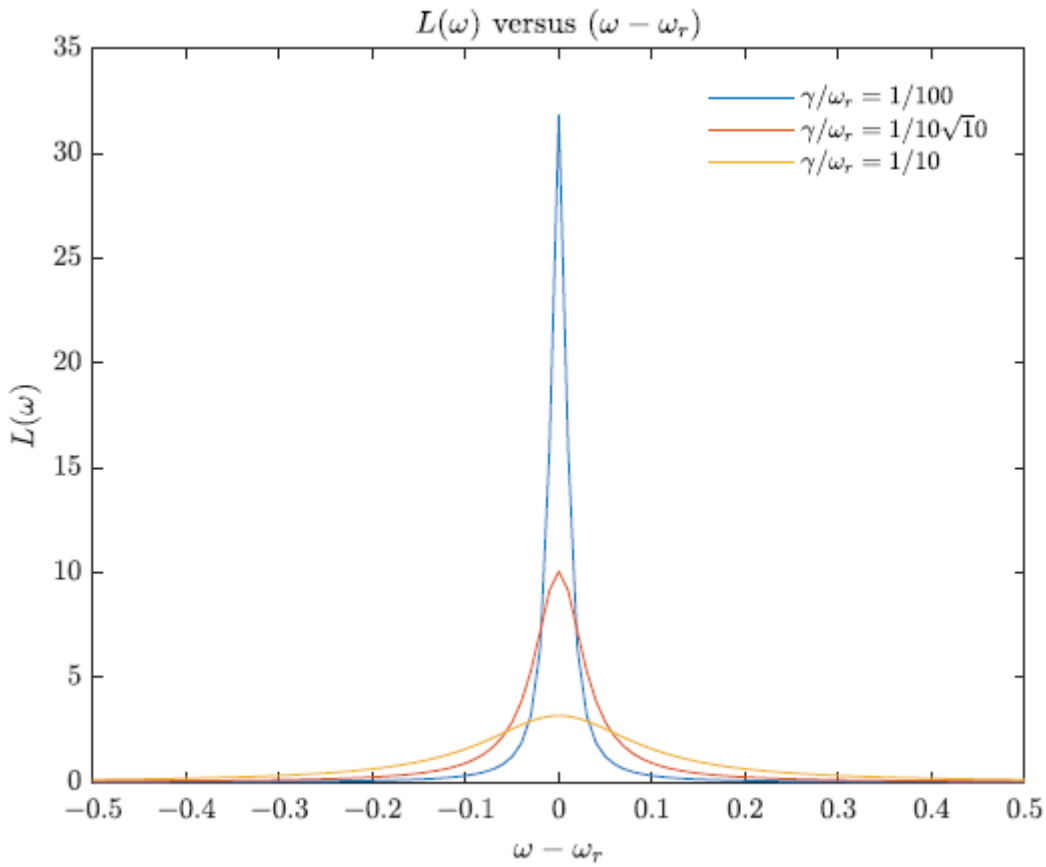

Figure 3: Solutions to Part B

2.3 Part C

2.3.1 Solving The Equations

This section is similar to Part B, except two different equations I.21 and I.28 are multiplied by γ/ω_r and plotted against $(\omega-\omega_r)/\gamma$. We are given the value for γ/ω_r this time as well. We can set a new variable $a=\gamma/\omega_r$, and upon picking an arbitrary value for ω_r, we can eliminate the unknown variable γ by using the known variables $a=\gamma/\omega_r$ and ω_r.

$$\text{I.21: } A_{\mathrm{D}} = \frac{\omega_r^2}{[(\omega_r^2-\omega^2)^2+4\omega^2\gamma^2]^{1/2}} \tag{12}$$

$$\text{I.28: } A_{\mathrm{D}}^2 = \frac{1}{4}\frac{\omega_r^2}{(\omega-\omega_r)^2+\gamma^2} \tag{13}$$

The goal is to plot $A_{\mathrm{D}}(\gamma/\omega_r)$ against $(\omega-\omega_r)/\gamma$. To do this, we need to express (12) and (13) as functions of $(\omega-\omega_r)/\gamma$ after being multiplied by γ/ω_r. The steps to do this for (12) are

$$\begin{aligned}
A_{\mathrm{D}}(\gamma/\omega_r) &= \left(\frac{\gamma}{\omega_r}\right)\frac{\omega_r^2}{[(\omega_r^2-\omega^2)^2+4\omega^2\gamma^2]^{1/2}} \\
&= \frac{\gamma\omega_r}{[(\omega_r^2-\omega^2)^2+4\omega^2\gamma^2]^{1/2}} \\
&= \frac{\omega_r}{(1/\gamma)[(\omega_r^2-\omega^2)^2+4\omega^2\gamma^2]^{1/2}} \\
&= \frac{\omega_r}{\{[(\omega^2-\omega_r^2)/\gamma]^2+4\omega^2\}^{1/2}} \\
&= \frac{\omega_r}{(\{[(\omega+\omega_r)(\omega-\omega_r)]/\gamma\}^2+4\omega^2)^{1/2}} \\
&= \frac{\omega_r}{[(\omega+\omega_r)^2[(\omega-\omega_r)/\gamma]^2+4\omega^2]^{1/2}}
\end{aligned} \tag{14}$$

(12) is now a function of $(\omega-\omega_r)/\gamma$, but there are still some ω terms that need to be removed. To this end, a new variable $x=(\omega-\omega_r)/\gamma$ is introduced. Solving the x equation for ω, we find that $\omega = x\gamma+\omega_r$. Plugging x and this solution for ω into the last equation in (14), we find that

$$A_{\mathrm{D}}(\gamma/\omega_r) = \frac{\omega_r}{[(x\gamma+2\omega_r)^2x^2+4(x\gamma+\omega_r)^2]^{1/2}} \tag{15}$$

Plugging $\gamma = a\omega_r$ into (15), we get

$$A_{\mathrm{D}}(\gamma/\omega_r) = \frac{\omega_r}{\{[\omega_r(xa+2)]^2x^2+4[\omega_r(xa+1)]^2\}^{1/2}} \tag{16}$$

For Eq. I.28 (13), the square root of both sides needs to be taken first to isolate A_{D}. Doing this gives

$$A_{\mathrm{D}} = \frac{1}{2}\frac{\omega_r}{2[(\omega-\omega_r)^2+\gamma^2]^{1/2}} \tag{17}$$

From here the steps to create the $A_{\mathrm{D}}(\gamma/\omega_r)$ against $(\omega-\omega_r)/\gamma$ relationship for (17) are

$$\begin{aligned} A_{\mathrm{D}}(\gamma/\omega_r) &= \left(\frac{\gamma}{\omega_r}\right)\frac{1}{2}\frac{\omega_r}{[(\omega-\omega_r)^2+\gamma^2]^{1/2}} \\ &= \frac{1}{2}\frac{\gamma}{[(\omega-\omega_r)^2+\gamma^2]^{1/2}} \\ &= \frac{1}{2}\frac{1}{(1/\gamma)[(\omega-\omega_r)^2+\gamma^2]^{1/2}} \\ &= \frac{1}{2}\frac{1}{\{[(\omega-\omega_r)/\gamma]^2+1\}^{1/2}} \end{aligned} \tag{18}$$

Similar to solving Eq. I.21 above, the stray ω term needs to be eliminated. Using the same variable $x = (\omega-\omega_r)/\gamma$ again, we get

$$A_{\mathrm{D}}(\gamma/\omega_r) = \frac{1}{2}\frac{1}{(x^2+1)^{1/2}} \tag{19}$$

Since (19) is only a function of x it does not depend on the value of a like the solution to Eq. I.21 does. This means that for the three different values of a that the problem asks be plotted, each plot will have the exact same behavior. This is a direct consequence of the simplification that is used when deriving Eq. I.28.

2.3.2 Plotting the Solutions

When Equations I.21 and I.28 are used in this problem, the multiplicative term γ/ω_r and reference variable $x = (\omega-\omega_r)/\gamma$ distort the previously Lorentzian form of the equations. By comparing how the distortion appears separately in both cases, conclusions can be drawn about how accurate the estimation equation I.28 is to the exact $A_{\mathrm{D}}(\gamma/\omega_r)$ equation I.21. The following MATLAB code and plots show the the plotting process for the solutions derived in Section 2.3.1 and are followed by a discussion of the results of the plots.

```
%Variable List
%g_omr_1 = first value of gamma / omega_r
%g_omr_2 = second value of gamma / omega_r
%g_omr_3 = third value of gamma / omega_r
%omega_r = arbitrary value of omega_r
%x = range of (omega - omega_r)/gamma for solution to be plotted against
%a = temporary value of g_omr_1/2/3 used in solution
%A_D_21_1 = A_D solution from Eq. I.21 with a set to g_omr_1
%A_D_21_2 = A_D solution from Eq. I.21 with a set to g_omr_2
%A_D_21_3 = A_D solution from Eq. I.21 with a set to g_omr_3

%Solution
g_omr_1 = 1/100;
g_omr_2 = 1/10;
g_omr_3 = 1/2;
%arbitrary value of omega_r, set to 1 for simplicity
omega_r = 1;
%range of x variable
x = -100:1/100:100;
%A_D_21_1/2/3 calculations
a = g_omr_1;
A_D_21_1 = omega_r ./ ((omega_r * (x*a + 2)).^2 .* (x.^2) + ...
    4*(omega_r * (x*a + 1)).^2).^0.5;
a = g_omr_2;
A_D_21_2 = omega_r ./ ((omega_r * (x*a + 2)).^2 .* (x.^2) + ...
    4*(omega_r * (x*a + 1)).^2).^0.5;
a = g_omr_3;
A_D_21_3 = omega_r ./ ((omega_r * (x*a + 2)).^2 .* (x.^2) + ...
    4*(omega_r * (x*a + 1)).^2).^0.5;
```

Figure 4: Code to Generate Equations for Eq. I.21 Solution

```
%Plot Solution for Eq. I.21
figure(2)
plot(x,[A_D_21_1; A_D_21_2; A_D_21_3])
%set x and y limits
xlim([-50 50])
ylim([0 0.7])
%set tick label interpreter to LaTeX
set(gca,'TickLabelInterpreter','LaTeX')
%label x and y axis and title
xlabel('$(\omega - \omega_{r})/\gamma$','interpreter','LaTeX')
ylabel('$A_{D}(\gamma / \omega)$','interpreter','LaTeX',...
    'fontname','Times New Roman')
title('$A_{D}(\gamma / \omega)$ versus $(\omega - \omega_{r})/\gamma$',...
    'interpreter','LaTex','fontname','Times New Roman')
%set x and y axis tick marks and tick labels
xticks([-50:10:50])
xticklabels({'$-50$','$-40$','$-30$','$-20$','$-10$','$0$',...
    '$10$','$20$','$30$','$40$','$50$'});
yticks([0:0.1:0.7])
yticklabels({'$0$','$0.1$','$0.2$','$0.3$','$0.4$','$0.5$',...
    '$0.6$','$0.7$'})
%turn on axis for all sides
box on
%set legend and remove box around it
lgd = legend('$a = 1/100$','$a = 1/10$','$a = 1/2$');
set(lgd,'interpreter','LaTeX')
legend('boxoff')
%save as pdf
saveas(gcf,'13c21.pdf','pdf')
```

Figure 5: Code to Plot Generated Equations for Eq. I.21 Solution

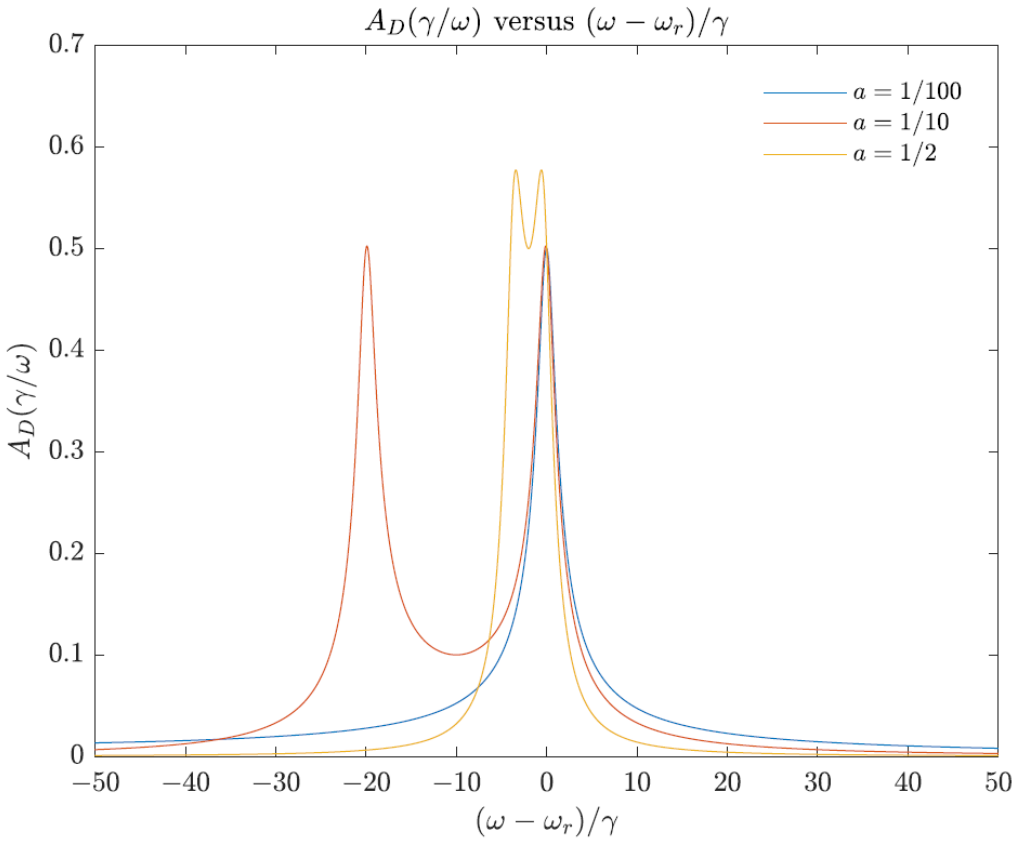

Figure 6: Solutions to Eq. I.21

```
%Variable List
%g_omr_1 = first value of gamma / omega_r
%g_omr_2 = second value of gamma / omega_r
%g_omr_3 = third value of gamma / omega_r
%omega_r = arbitrary value of omega_r
%x = range of (omega - omega_r)/gamma for solution to be plotted against
%a = temporary value of g_omr_1/2/3 used in solution
%A_D_28_1 = A_D solution from Eq. I.28 with a set to g_omr_1
%A_D_28_2 = A_D solution from Eq. I.28 with a set to g_omr_2
%A_D_28_3 = A_D solution from Eq. I.28 with a set to g_omr_3

%Solution
g_omr_1 = 1/100;
g_omr_2 = 1/10;
g_omr_3 = 1/2;
%arbitrary value of omega_r, set to 1 for simplicity
omega_r = 1;
%range of x variable
x = -100:1/100:100;
%A_D_28_1/2/3 calculations
a = g_omr_1;
A_D_28_1 = 1 ./ (2 * (x.^2 + 1).^.5);
a = g_omr_2;
A_D_28_2 = 1 ./ (2 * (x.^2 + 1).^.5);
a = g_omr_3;
A_D_28_3 = 1 ./ (2 * (x.^2 + 1).^.5);
```

Figure 7: Code to Generate Equations for Eq. I.28 Solution

```
%Plot Solution for Eq. I.28
figure(3)
plot(x,[A_D_28_1; A_D_28_2; A_D_28_3])
%set x and y limits
xlim([-50 50])
ylim([0 0.7])
%set tick label interpreter to LaTeX
set(gca,'TickLabelInterpreter','LaTeX')
%label x and y axis and title
xlabel('$(\omega - \omega_{r})/\gamma$','interpreter','LaTeX')
ylabel('$A_{D}(\gamma / \omega)$','interpreter','LaTeX',...
    'fontname','Times New Roman')
title('$A_{D}(\gamma / \omega)$ versus $(\omega - \omega_{r})/\gamma$',...
    'interpreter','LaTex','fontname','Times New Roman')
%set x and y axis tick marks and tick labels
xticks([-50:10:50])
xticklabels({'$-50$','$-40$','$-30$','$-20$','$-10$','$0$',...
    '$10$','$20$','$30$','$40$','$50$'});
yticks([0:0.1:0.7])
yticklabels({'$0$','$0.1$','$0.2$','$0.3$','$0.4$','$0.5$',...
    '$0.6$','$0.7$'})
%turn on axis for all sides
box on
%set legend and remove box around it
lgd = legend('$a = 1/100$','$a = 1/10$','$a = 1/2$');
set(lgd,'interpreter','LaTeX')
legend('boxoff')
%save as pdf
saveas(gcf,'13c28.pdf','pdf')
```

Figure 8: Code to Plot Generated Equations for Eq. I.28 Solution

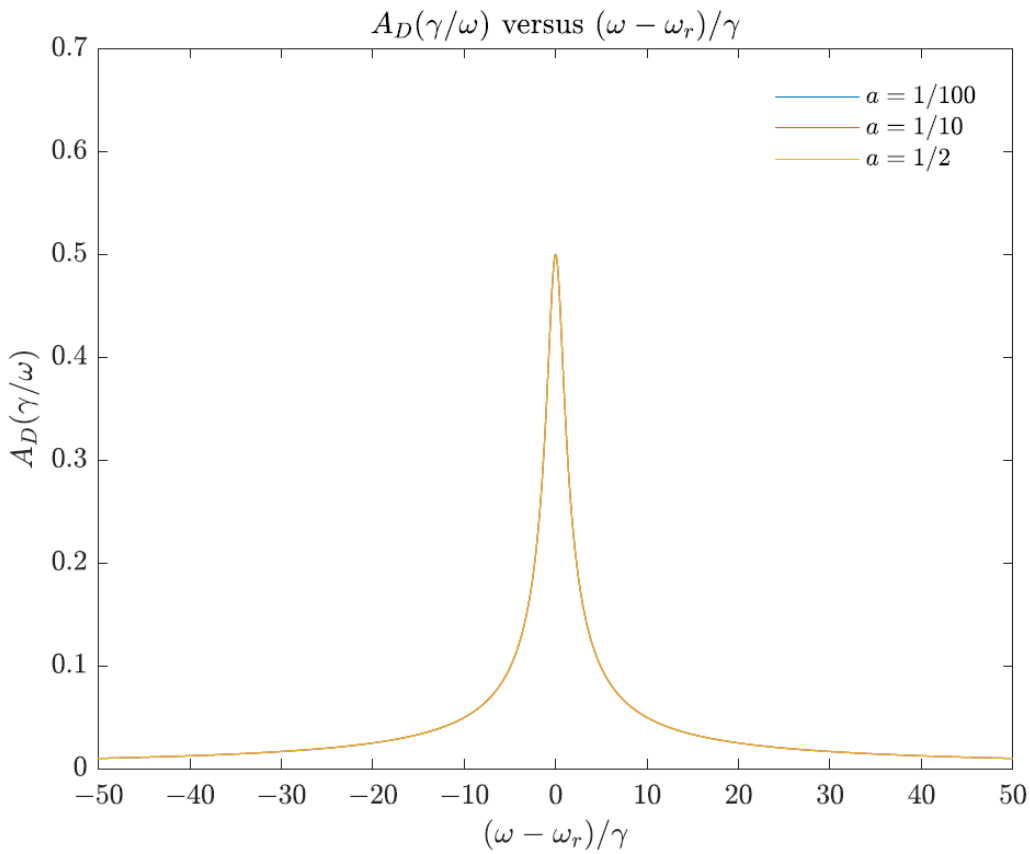

Figure 9: Solutions to Eq. I.28

2.3.3 Solution Analysis

Eq. I.21 As the variable a increases, the γ value increases as well since ω_r is assumed to be an arbitrary value. This change is reflected in the behavior of each of the three solutions plotted. As the γ value increases, the width of the curve around the resonant frequency increases as well, as shown in Figure 6 by the curves' behavior around $(\omega - \omega_r)/\gamma = 0$. Additionally, each curve has a peak at frequencies lower than their resonant frequency at very specific locations. In general, each curve will reach a peak above $A_{\mathrm{D}} = 0.5$ at both $(\omega - \omega_r)/\gamma = -\gamma$ and $(\omega - \omega_r)/\gamma = -2/\gamma$, while equaling 0.5 at $(\omega - \omega_r)/\gamma = -\gamma \pm \gamma$ and $(\omega - \omega_r)/\gamma = -2/\gamma \pm \gamma$. The larger the value of γ, the higher above 0.5 the peak will reach. At 0 on the x-axis, each curve hits 0.5 and then progresses to the right without any other distortions.

The cause of this specific behavior can be seen in the last step of Equation 14. As the respective term $(\omega - \omega_r)/\gamma$ is isolated, a separate $(\omega + \omega_r)^2$ term gets pulled out. This term creates specific behaviors while operating to the negative of the resonance frequency, causing the peaks observed in the previous paragraph. In particular, attention needs to be paid towards how the γ term in the denominator of x warps the x axis. Because of this warping, the first curve with the lowest γ value has a much larger x-axis warping than the other curves, scaled by the proportional difference in γ value. When considered in whole, the specific set of conditions used in this problem create a normalized set of solutions that are scaled specifically by differences in the γ value used.

Eq. I.28 The behavior of the solutions of Eq. I.28 are as mentioned above, at the end of Section 2.3.1. Since the solution is only dependent on x, there is no deviation as the value of γ is increasing. Due to some of the assumptions that Eq. I.28 relies on there should be a decrease in accuracy of the estimate. However, the multiplication of A_{D} by γ/ω and the γ factor in the denominator of x both correct the solution to Eq. I.28 to ensure that it stays unchanged.

When compared under the warping imposed by the problem, Eq. I.28 is not a very good estimate of Eq. I.21 until γ reaches a sufficiently low value. However, this is to expected, as one of the critical assumptions of Eq. I.28 is that only cases where $\omega_r/\gamma \gg 1$, so that in situations where that assumption becomes more true the estimated equation gets more accurate. Additionally, the specific solution path here chose to focus on the shift in γ values as a changed, intentionally isolating γ and not ω_r. A different solution approach that isolated ω_r instead would see similar changes as ω_r increased as were seen here when γ decreased. This behavior shows that as the oscillator's properties improve in quality, models from Eq. I.28 improve in accuracy to I.21. From observations of the behaviors of the solutions from this problem, one can conclude that Eq. I.28 is a good estimate of Eq. I.21 as long as the oscillator can be considered a good oscillator.